Statistics - Homework 6 (Due May 28, 2021)

1. Let X be a gamma random variable with the following density:
   $f_X(x;\alpha,\beta) = \frac{1}{\alpha\Gamma(\beta)}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-(x/\alpha)}, \quad 0 \le x < +\infty. \quad \alpha = 40, \beta = 2.$
   (1) Generate 10,000 sets of random samples, each of sample size n=10.
   (2) Calculate the maximum likelihood estimates (MLE) of the two parameters α and β for each of the 10,000 random samples.
   (3) Calculate the mean of the MLE of α and β, and compare the values against α and β.
   (4) Plot the frequency histogram of the maximum likelihood estimates.
2. Let $(x_1, x_2, \cdots, x_n)$ be a random sample of size n from a distribution with a cumulative distribution function $F_X(\cdot)$ and $x_1 < x_2 < \cdots < x_n$. The empirical cumulative distribution function (ECDF) based on the random sample is defined as
   $$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x]}(x_i); \quad -\infty < x < \infty.$$
   Prove the following results:
   (1) $F_n(X)$ is asymptotically normal, i.e., $F_n(X)$ approaches a normal distribution as n approaches infinity.
   (2) $E[F_n(X)] = F_X(X)$
   (3) $Var[F_n(X)] = \frac{1}{n}F_X(X)[1-F_X(X)]$
   (4) $\sqrt{n}[F_n(X) - F_X(X)]$ approaches a normal distribution with mean 0 and variance $F_X(X)[1-F_X(X)]$ as n approaches infinity.
3. A random sample of size n (i.e., $x_1, x_2, \cdots, x_n$) from an exponential density $f_X(x;\ \lambda) = \lambda e^{-\lambda x}$ is available.
   (1) Prove that the maximum likelihood estimate of λ is $\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}_n}$.
   (2) Prove that the above maximum likelihood estimator is (or is not) an unbiased estimator. *Hint*: The sample mean of a random sample from an exponential density has a gamma distribution. [Note: $E\left[\frac{1}{X}\right] = \frac{1}{E[X]}$ is not generally true.]
4. A random sample of size n_1 from a normal distribution with mean μ_1 and standard deviation σ_1 is available. Another random sample of size n_2 from a normal distribution with mean μ_2 and standard deviation σ_2 is also available. The two random samples are independent.
   (1) Find a maximum likelihood estimator of $\theta = \mu_1 - \mu_2$.
   (2) If the sum of n_1 and n_1 $(n = n_1 + n_2)$ is fixed, how should the sizes of the two random samples be chosen in order to minimize the variance of the maximum likelihood estimator in (1).
5. Let $(x_1, x_2, \cdots, x_n)$ be a random sample from an exponential density $f_X(x;\ \lambda) = \lambda e^{-\lambda x}$ and $(y_1, y_2, \cdots, y_m)$ be a random sample from another exponential density $f_Y(y;\ \theta) = \theta e^{-\theta y}$ with $\theta = 2\lambda$. The two random samples are independently observed. Find the maximum likelihood estimate of λ.
6. Let $X_1, X_2, \cdots, X_n$ be independently and identically distributed (IID) from a

continuous uniform distribution $U[0,\ \theta]$.

(1) Find a maximum likelihood estimator of θ, and then find the mean and mean-squared error of your estimator.

(2) Find a method of moments estimator of θ, and then find the mean and mean-squared error of your estimator.

7. Let $(x_1,\ x_2,\cdots,x_n)$ be a random sample from a distribution with probability density $f_X(x;\ \theta)=\dfrac{\theta 2^{\theta}}{x^{\theta+1}},\ x\geq 2,\ \theta>1$.

(1) Find the method of moments estimate of θ.

(2) Find the maximum likelihood estimate of θ.

8. The radius (r) of a circle is measured with an error of measurement which is normally distributed with zero mean and unknown variance σ^2. Given the following independent measurements of the radius:

51.07	52.49	48.07	48.94	49.33
50.34	51.30	40.33	41.38	44.05
51.45	48.89	56.42	49.21	49.41
46.03	53.64	41.13	46.17	57.07

Find an unbiased estimator of the area of the circle and use the above data to calculate an unbiased estimate of the circle area.