GEOSTATISTICS

Gamma Random Field Simulation

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- **A *random field* can be defined as a set of jointly distributed random variables defined in a spatial domain (2-, 3-, or higher dimension).**
- **Examples of random fields**
  - Spatial variation of rainfall
  - Variation of terrain elevation
  - Spatial variation of heavy metal contamination
  - Grey level (reflectance) of multispectral images

Conceptual description of a gamma random field simulation approach

Given a pair of bivariate gamma random variables (X,Y) with know properties $\mu_X, \mu_Y, \gamma_X, \gamma_Y, \rho_{XY}$

↓

Converting ρ_{XY} to ρ_{UV} where (U,V) represents a pair of bivariate standard normal variables.

↓

Given a homogeneous and isotropic random field $Z(x)$ with known gamma density and covariance function $C_Z(h)$ or variogram $\gamma_Z(h)$.

↓

Converting $C_Z(h)$ to $C_W(h)$ where $W(x)$ is a random field with standard normal density and covariance function $C_W(h)$.

↓

Generating a random sample of (U,V) with sample size n, i.e. $\{(u_i, v_i), i = 1, \cdots, n\}$.

Generating a realization of W, i.e. $\{w(i, j), i = 1, \cdots, n \,;\, j = 1, \cdots, m\}$ where (i,j) represents a spatial location and n and m defines the extent of the spatial domain.

Individually and independently converting u_i to x_i and v_i to y_i. The resultant $\{(x_i, y_i), i = 1, \cdots, n\}$ is a random sample of the bivariate random variables (X,Y).

Individually and independently converting $w(i, j)$ to $z(i, j)$. The resultant $\{z(i, j), i = 1, \cdots, n \,;\, j = 1, \cdots, m\}$ is a realization of the random field $Z(x)$.

Components of sequential random field simulation

(1) Converting the covariance function $C_Z(h)$ of a gamma random field $Z(x)$ to the covariance function $C_W(h)$ of a corresponding Gaussian random field $W(x)$. In a sequential random field simulation process, the covariance function appears as a covariance matrix Σ which involves a point for random number generation and its neighboring points. Thus, conversion between the covariance function $C_Z(h)$ and $C_W(h)$ is equivalent to conversion between the covariance matrices Σ_Z and Σ_W.

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(2) Generating realizations of the Gaussian random field with covariance function $C_W(h)$, and

(3) Transforming realizations of $W(x)$ to corresponding realizations of the gamma random field $Z(x)$.

Sequential Gaussian simulation

Given a homogeneous and isotropic Gaussian random field $\{Z(x), x \in \Omega\}$ with known probability density function $f_Z(z)$ and covariance function $C_Z(h)$ (or semi-variogram $\gamma_Z(h)$), we want to generate as many random samples (realizations) of the random field.

Bivariate Normal Distribution

- **Bivariate normal density function**

$$f_{XY}(x,y) = f_Z(z) = \frac{1}{2\pi\left|\Sigma\right|^{1/2}} e^{-\frac{1}{2}\left[(z-\mu)^T \Sigma^{-1} (z-\mu)\right]}$$

where $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ and $\mu = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$

are respectively the mean vector and covariance matrix of

$Z = \begin{pmatrix} X \\ Y \end{pmatrix}$.

Multivariate Normal Distribution

For a set of $(p+q)$ normally distributed random variables, say $W' = (W_1, \cdots, W_{p+q})$, the multivariate joint density is given by (Morrison, 1990)

$$f_W(w) = \frac{1}{(2\pi)^{(p+q)/2} \, |\Sigma_W|^{1/2}} e^{-\frac{1}{2}(w-\mu)'\Sigma_W^{-1}(w-\mu)}$$

where Σ_W is the covariance matrix with $(p+q)\times(p+q)$ dimension and μ is the $(p+q)$ dimensional mean vector.

Let W be divided into two subsets $W_1' = (W_1, \cdots, W_p)$ and $W_2' = (W_{p+1}, \cdots, W_{p+q})$, the mean vector and the covariance matrix can thus be respectively expressed as

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \Sigma_W = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix}$$

where μ_1 and μ_2 are respectively the mean vectors of W_1 and W_2, and Σ_{ij} is the covariance matrix of W_i and W_j $(i, j = 1 \text{ or } 2)$.

Conditional normal density

- **Conditional normal density** $f_{Y|X=x}(y)$

$$f_{Y|X}(Y=y\,|\,x)=\frac{1}{\sqrt{2\pi(1-\rho^2)}\cdot\sigma_Y}\cdot\exp\left\{-\frac{1}{2}\left[\frac{(y-\mu_Y)-\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)}{\sigma_Y\sqrt{1-\rho^2}}\right]^2\right\}$$

Conditional multivariate normal density

Suppose that values of W_2 are known, i.e. $w_2' = (w_{p+1}, \cdots, w_{p+q})$, the conditional multivariate density of W_1 given $W_2 = w_2$ can then be expressed by

$$f_{W_1|W_2}(w_1 \mid w_2) = \frac{1}{(2\pi)^{p/2} |\Sigma^*|^{1/2}} e^{-\frac{1}{2}(w_1 - \mu^*)'(\Sigma^*)^{-1}(w_1 - \mu^*)} \quad \textbf{[C]}$$

$$\mu^* = \mu_{W_1|w_2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(w_2 - \mu_2)$$

$$\Sigma^* = \Sigma_{W_1|w_2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}'$$

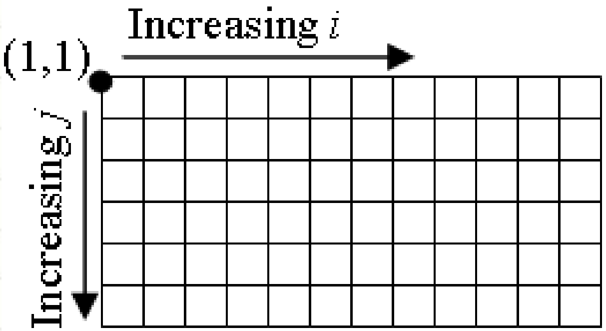
- **Equation [C] lays the foundation for stochastic simulation of a Gaussian random field.**
- **Random field simulation is generally carried out by sequentially generating random number at only one target location each time.**

Thus, for sequential stochastic simulation of a standard Gaussian random field, μ_1 and Σ_{11} respectively reduce to 0 and 1, and the covariance matrix becomes

$$\Sigma_W = \left[\begin{array}{c|ccc} C_{11} & C_{12} & \cdots & C_{1,1+q} \\ \hline C_{21} & C_{22} & \cdots & C_{2,1+q} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1+q,1} & C_{1+q,2} & \cdots & C_{1+q,1+q} \end{array}\right] = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{12}' & \Sigma_{22} \end{array}\right]$$

where C_{ij} represents the covariance between two univariate random variables W_i and W_j, and $C_{ii} = 1$ for all $i = 1, \cdots, 1+q$.

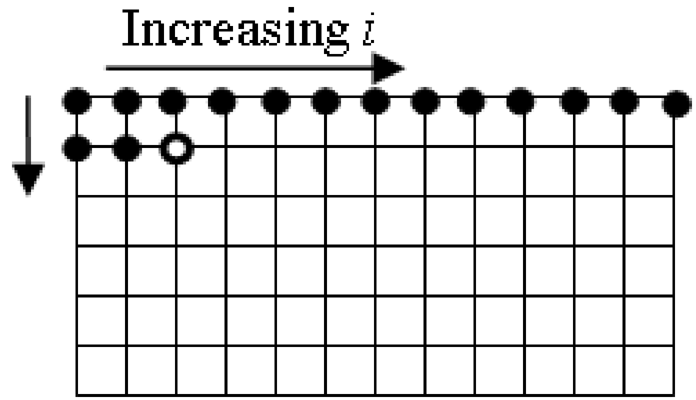
- Suppose the random field simulation begins with a univariate random number generation at an initial point x_0 with coordinate (1,1).



(a) Univariate standard normal simulation at the starting point (1,1).

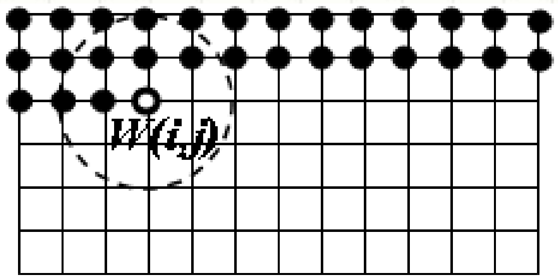
- We then sequentially generate random numbers at other locations under the condition of previously generated random numbers.

- **The simulation is conducted following a column-preference style in which random numbers at all nodes of the same line are generated sequentially and then the process proceeds to the next line.**

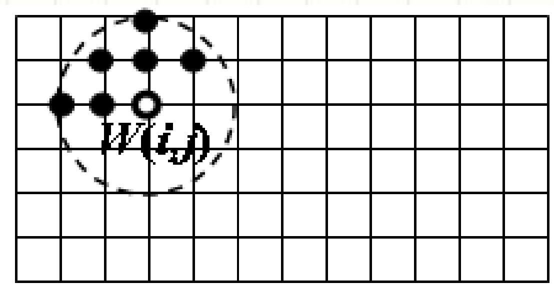


(b) Conditional Gaussian simulation. Simulation is conducted by a column-preference style.

- **At any stage of the simulation process, the number and locations of the conditioning variates depend on the range measured in terms of the grid interval.**



(c) Determination of conditioning variates for conditional simulation of $W(i,j)$ using the influence range.



(d) Conditioning variates (marked by solid dots) to be included in conditional simulation of $W(i,j)$.

Finally, the covariance matrices Σ_{kl} $(k, l = 1 \text{ or } 2)$ and Σ^* can be established using the given semi-variogram $\gamma(h)$ or covariance function $C(h)$ of the random field.

In our random field simulation approach, the covariance matrix Σ_W is obtained by transforming from the covariance matrix of a corresponding gamma random field to ensure the resultant realizations of the gamma random field having desired statistical properties.

$$f_{W_1|W_2}(w_1 \mid w_2)$$

$$= \frac{1}{(2\pi)^{p/2} \mid \Sigma^* \mid^{1/2}} e^{-\frac{1}{2}(w_1-\mu^*)'(\Sigma^*)^{-1}(w_1-\mu^*)} \quad \textbf{[C]}$$

$$\mu^* = \mu_{W_1|w_2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(w_2 - \mu_2)$$

$$\Sigma^* = \Sigma_{W_1|w_2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}'$$

$$\Sigma_W = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1,1+q} \\ C_{21} & C_{22} & \cdots & C_{2,1+q} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1+q,1} & C_{1+q,2} & \cdots & C_{1+q,1+q} \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix}$$

Covariance matrices conversion

Until now we have shown details for stochastic simulation of a standard Gaussian random field $W(x)$ with covariance function $C_W(h)$ or variogram $\gamma_W(h)$.

However, our objective is to generate realizations of a gamma random field $Z(x)$ with a desired probability density function $f_Z(z)$ and a given covariance function $C_Z(h)$ or variogram $\gamma_Z(h)$.

A gamma-field counter part of Σ_W can be expressed by

$$\Sigma_Z = \begin{bmatrix} \Sigma_{11}^{G} & \Sigma_{12}^{G} \\ \Sigma_{12}^{G'} & \Sigma_{22}^{G} \end{bmatrix}$$

We have implicitly assumed that the covariance function $C_W(h)$ is given or known.

This unfortunately raises a question of how can we be sure that our stochastic simulation process using the assumed $C_W(h)$ will eventually yield realizations of $Z(x)$ which are associated with the desired covariance function $C_Z(h)$.

Apparently, the covariance function $C_W(h)$ cannot be arbitrarily chosen and a conversion between $C_W(h)$ and $C_Z(h)$ (or between Σ_Z and Σ_W) is necessitated.

$\rho_{XY} \sim \rho_{UV}$ Conversion

$$\rho_{XY} \approx \left(A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y\right)\rho_{UV}$$

$$+ 2B_X B_Y \rho_{UV}^2 + 6C_X C_Y \rho_{UV}^3$$ **[B]**

$$A_X = 1 + \left(\frac{\gamma_X}{6}\right)^4 \quad B_X = \frac{\gamma_X}{6} - \left(\frac{\gamma_X}{6}\right)^3 \quad C_X = \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^2$$

$$A_Y = 1 + \left(\frac{\gamma_Y}{6}\right)^4 \quad B_Y = \frac{\gamma_Y}{6} - \left(\frac{\gamma_Y}{6}\right)^3 \quad C_Y = \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^2$$

Transforming Gaussian realizations to gamma realizations – Variable transformation

Let X be a gamma random variable, i,e, $X \sim G(\alpha, \lambda)$, with the following density

$$f_X(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad \alpha, \lambda > 0 \text{ and } \ 0 \le x < +\infty.$$

We define a new random variable $Y = g(X) = 2\lambda X$ and the probability density function of Y can be derived by transformation of random variables.

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$$

$$f_Y(y) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}\left(\frac{y}{2\lambda}\right)^{\alpha-1} e^{-\lambda y/(2\lambda)} \frac{1}{2\lambda}$$

$$= \frac{1}{\Gamma(\alpha)2^{\alpha}} y^{\alpha-1} e^{-y/2}$$

The above equation is a chi-squared density with degree of freedom $k = 2\alpha$, i.e. $Y \sim \chi^2(k)$.

- **An approximation of the chi-squared distribution by the standard normal distribution, known as the *Wilson-Hilferty approximation*, is given as follows (Patel and Read, 1996)**

$$y \approx 2\alpha\left\{1-\frac{1}{9\alpha}+w\sqrt{\frac{1}{9\alpha}}\right\}^3$$

where *w* represents the standard normal deviate and *y* is the corresponding chi-squared variate.

- **Transformation of the standard normal deviate *w* to gamma variate *x* can thus be derived as**

$$x = \frac{y}{2\lambda} \approx \frac{\alpha}{\lambda}\left\{1 - \frac{1}{9\alpha} + w\sqrt{\frac{1}{9\alpha}}\right\}^3 \qquad \textbf{[D]}$$

- **Equation [D] is a one-to-one mapping function between *x* and *w*.**

Transforming Gaussian realizations to gamma realizations – Using frequency factor

General equation of frequency analysis: $X = \mu + K\sigma$
***K*: the frequency factor, a random variable**

$$f_X(x) = \frac{1}{\alpha\Gamma(\beta)}\left(\frac{x-\varepsilon}{\alpha}\right)^{\beta-1} e^{-[(x-\varepsilon)/\alpha]},$$

$$\varepsilon \le x < +\infty$$

$$\alpha = \sigma/\sqrt{\beta},\ \beta = (2/\gamma)^2,\ \varepsilon = \mu - \sigma\sqrt{\beta}$$

μ, σ and γ are respectively the mean, standard deviation and skewness coefficient of X

$$K_T \approx z + (z^2-1)\frac{\gamma}{6} + \frac{1}{3}(z^3-6z)\left(\frac{\gamma}{6}\right)^2 - (z^2-1)\left(\frac{\gamma}{6}\right)^3 + z\left(\frac{\gamma}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma}{6}\right)^5$$

z: standard normal deviate with exceedance probability $1/T$

Table I. Probability density functions and frequency factors of distributions commonly used for hydrological frequency analysis

Distribution, X	Probability density function $f_X(x)$	Frequency factor K_T
Normal	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$, $-\infty < x < +\infty$	Standard normal deviate z with exceedance probability $1/T$
Log–normal	$f_X(x) = \frac{1}{\sqrt{2\pi} x \sigma_y} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$, $0 < x < +\infty$ $\mu = e^{\mu_y + \sigma_y^2/2}$, $\sigma^2 = (e^{\sigma_y^2} - 1)\mu^2$ μ_y and σ_y are respectively the mean and standard deviation of $Y = \ln X$	$K_T = \frac{\exp\{[\ln(1+C_V^2)]^{1/2} Z - [\ln(1+C_V^2)]/2\} - 1}{C_V}$ $C_V = \sigma/\mu$, coefficient of variation of X z: standard normal deviate with exceedance probability $1/T$
EV1	$f_X(x) = \alpha \exp[-\alpha(x-\beta) - e^{-\alpha(x-\beta)}]$, $-\infty < x < +\infty$ $\alpha = \pi/\sqrt{6}\sigma$, $\beta = \mu - (0{\cdot}5772/\alpha)$ μ and σ are respectively the mean and standard deviation of X	$K_T = -\frac{\sqrt{6}}{\pi}\left\{0{\cdot}5772 + \ln\left[\ln\left(\frac{T}{T-1}\right)\right]\right\}$
PT3	$f_X(x) = \frac{1}{\alpha\Gamma(\beta)}\left(\frac{x-\varepsilon}{\alpha}\right)^{\beta-1} e^{-[(x-\varepsilon)/\alpha]}$, $\varepsilon \le x < +\infty$ $\alpha = \sigma/\sqrt{\beta}$, $\beta = (2/\gamma)^2$, $\varepsilon = \mu - \sigma\sqrt{\beta}$ μ, σ and γ are respectively the mean, standard deviation and skewness coefficient of X	$K_T \approx z + (z^2-1)\frac{\gamma}{6} + \frac{1}{3}(z^3-6z)\left(\frac{\gamma}{6}\right)^2 - (z^2-1)\left(\frac{\gamma}{6}\right)^3 + z\left(\frac{\gamma}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma}{6}\right)^5$ z: standard normal deviate with exceedance probability $1/T$
LPT3	$f_X(x) = \frac{1}{\alpha x\Gamma(\beta)}\left(\frac{\ln x-\varepsilon}{\alpha}\right)^{\beta-1} e^{-\left(\frac{\ln x-\varepsilon}{\alpha}\right)}$, $\varepsilon \le \ln x < +\infty$ $\alpha = \sigma_y/\sqrt{\beta}$, $\beta = (2/\gamma_y)^2$, $\varepsilon = \mu_y - \sigma_y\sqrt{\beta}$ μ_y, σ_y and γ_y are respectively the mean, standard deviation and skewness coefficient of $Y = \ln X$	Same as K_T of the PT3 distribution (K_T is to be substituted into $y_T = \ln x_T = \mu_y + K_T\sigma_y$)

Summary of the simulation procedures

(1) Generate a standard Gaussian random number at the initial node (1,1),

(2) Determine neighboring nodes to be involved in the subsequent simulation by considering range of the random field $Z(x)$, and use the covariance function $C_Z(h)$ to establish the covariance matrix Σ_Z ,

(3) Transform Σ_Z to Σ_W using Equation [B],

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(4) Generate a standard Gaussian number at the target node using the conditional Gaussian density of Equation [C],

(5) Repeat procedures (2) to (4) until a realization of the standard Gaussian random field is established,

(6) Perform point-to-point Gaussian-to-gamma transformation using Equation [D], and it yields a realization of the gamma random field with desired properties.

$\rho_{XY} \sim \rho_{UV}$ Conversion

$$\rho_{XY} \approx \left(A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y\right)\rho_{UV}$$

$$+ 2B_X B_Y \rho_{UV}^2 + 6C_X C_Y \rho_{UV}^3 \qquad \textbf{[B]}$$

$$A_X = 1 + \left(\frac{\gamma_X}{6}\right)^4 \qquad B_X = \frac{\gamma_X}{6} - \left(\frac{\gamma_X}{6}\right)^3 \qquad C_X = \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^2$$

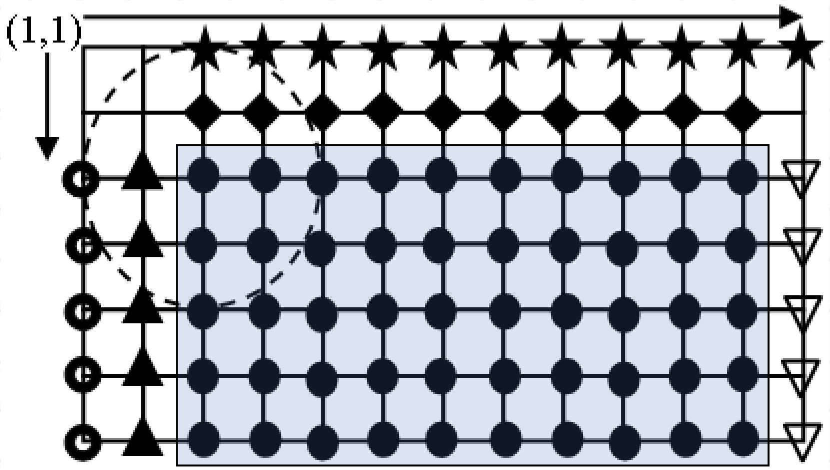
$$A_Y = 1 + \left(\frac{\gamma_Y}{6}\right)^4 \qquad B_Y = \frac{\gamma_Y}{6} - \left(\frac{\gamma_Y}{6}\right)^3 \qquad C_Y = \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^2$$

$$f_{W_1|W_2}(w_1 \mid w_2) = \frac{1}{(2\pi)^{p/2} |\Sigma^*|^{1/2}} e^{-\frac{1}{2}(w_1-\mu^*)'(\Sigma^*)^{-1}(w_1-\mu^*)} \quad \textbf{[C]}$$

$$x = \frac{y}{2\lambda} \approx \frac{\alpha}{\lambda}\left\{1 - \frac{1}{9\alpha} + w\sqrt{\frac{1}{9\alpha}}\right\}^3 \quad \textbf{[D]}$$

It's worthy to note that in implementing the above procedures a common covariance matrix Σ_Z or Σ_W can be used at nodes with the same geometric pattern of the target node and its conditioning nodes.

Utilizing this property helps to reduce the execution time for random field simulation.



- **Illustration of nodes with common covariance matrix for conditional simulation using a column-preference generation algorithm. Nodes marked by the same symbols have a common covariance matrix Σ_W . (The range is assumed to be twice of grid interval.)**

Simulation and verification

Parameters of the gamma density and spherical semi-variogram model designated for random field stochastic simulation.

Scenario type	Gamma density parameters μ	γ	α	λ	Variogram parameters $\omega=\sigma^2$	a	Size of simulation†
I	0.67	2.985	0.449	0.67	1	1, 2, 3, 6	80×80 40×40
II	1	2	1	1	1	1, 2, 3, 6	80×80 40×40
III	2	1	4	2	1	1, 2, 3, 6	80×80 40×40
IV	4	0.5	16	4	1	1, 2, 3, 6	80×80 40×40

†Size of simulation represents the spatial domain Ω of the random field $Z(x)$.

- **Since our simulation is based on a discrete network of one pixel grid interval, the random field with one-pixel range is technically completely random with no spatial correlation between any neighboring pixels, resulting in a pure nugget semi-variogram.**
- **As the range increases, the degree of spatial correlation increases.**
- **One hundred simulation runs were conducted for each scenario type with respect to specific values of range and simulation size**

- **For a given scenario type and a specific value of range, parameter estimation was done for each of the 100 realizations.**
- **These estimates vary among different realizations. Therefore, the sample mean and standard deviation of these realization-specific parameter estimates were calculated.**

Scenario I Size of simulation 80×80 ($\mu = 0.67, \gamma = 2.985, \omega = \sigma^2 = 1$)

Parameter Estimator	summary statistics	a (range of the semi-variogram) 1	2	3	6
$\hat{\mu}_Z$	mean	0.6611	0.6595	0.6593	0.6771
	std dev	0.0138	0.0202	0.0375	0.0889
	CV	0.0208	0.0307	0.0569	0.1314
$\hat{\sigma}_Z$	mean	0.9930	0.9738	0.9874	1.0286
	std dev	0.0542	0.0758	0.1004	0.2229
	CV	0.0546	0.0779	0.1017	0.2167
$\hat{\gamma}_Z$	mean	3.1958	3.0818	3.1665	2.9594
	std dev	0.3647	0.2408	0.3156	0.3702
	CV	0.1141	0.0781	0.0997	0.1251
$\hat{\omega}$	mean	0.9933	0.9735	0.9886	1.0496
	std dev	0.0569	0.0787	0.1080	0.2421
	CV	0.0573	0.0808	0.1093	0.2307
$\hat{a}$	mean	NA	2.1162	3.1601	6.1801
	std dev	NA	0.0852	0.1425	0.4537
	CV	NA	0.0403	0.0451	0.0734

Size of simulation 40×40 ($\mu = 0.67, \gamma = 2.985, \omega = \sigma^2 = 1$)

Parameter Estimator	summary statistics	a (range of the semi-variogram)			
		1	2	3	6
$\hat{\mu}_Z$	mean	0.6613	0.6579	0.6564	0.6504
	std dev	0.0261	0.0345	0.0739	0.1783
	CV	0.0395	0.0524	0.1126	0.2741
$\hat{\sigma}_Z$	mean	0.9836	0.9745	0.9746	0.9257
	std dev	0.1040	0.1203	0.2047	0.5135
	CV	0.1057	0.1235	0.2100	0.5547
$\hat{\gamma}_Z$	mean	3.1327	3.1175	3.0835	2.7188
	std dev	0.5499	0.4241	0.6299	0.5613
	CV	0.1755	0.1360	0.2043	0.2064
$\hat{\omega}$	mean	0.9824	0.9781	0.9775	0.9518
	std dev	0.1064	0.1262	0.2147	0.5690
	CV	0.1083	0.1290	0.2196	0.5978
$\hat{a}$	mean	NA	2.0692	3.1096	5.7735
	std dev	NA	0.1295	0.2445	0.5304
	CV	NA	0.0626	0.0786	0.0919

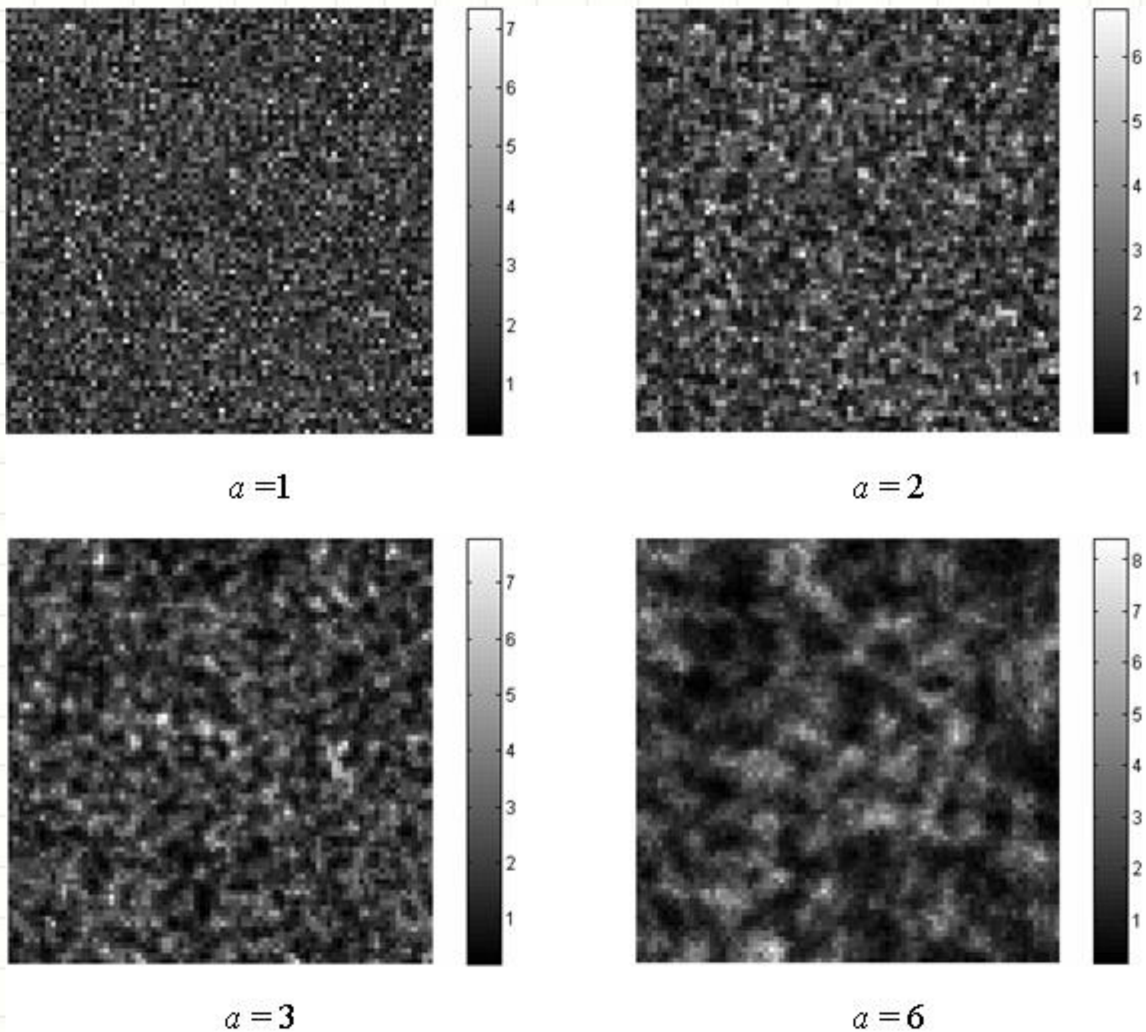
Scenario IV. Size of simulation 80×80 ($\mu = 4, \gamma = 0.5, \omega = \sigma^2 = 1$)

Parameter Estimator	summary statistics	a (range of the semi-variogram)			
		1	2	3	6
$\hat{\mu}_Z$	mean	4.0039	3.9979	4.0007	4.0191
	std dev	0.0124	0.0183	0.0328	0.0533
	CV	0.0031	0.0046	0.0082	0.0133
$\hat{\sigma}_Z$	mean	0.9998	0.9978	1.0027	1.0076
	std dev	0.0204	0.0220	0.0326	0.0643
	CV	0.0204	0.0220	0.0325	0.0638
$\hat{\gamma}_Z$	mean	0.5042	0.5004	0.5018	0.5078
	std dev	0.0339	0.0366	0.0487	0.0992
	CV	0.0673	0.0732	0.0970	0.1953
$\hat{\omega}$	mean	1.0005	0.9970	1.0027	1.0153
	std dev	0.0210	0.0228	0.0332	0.0691
	CV	0.0210	0.0229	0.0331	0.0680
$\hat{a}$	mean	NA	2.0101	3.0412	5.9864
	std dev	NA	0.0437	0.0708	0.2672
	CV	NA	0.0218	0.0233	0.0446

Size of simulation 40×40 ($\mu = 4, \gamma = 0.5, \omega = \sigma^2 = 1$)

Parameter Estimator	summary statistics	a (range of the semi-variogram) 1	2	3	6
$\hat{\mu}_Z$	mean	4.0001	3.9978	4.0075	4.0026
	std dev	0.0244	0.0475	0.0573	0.1685
	CV	0.0061	0.0119	0.0143	0.0421
$\hat{\sigma}_Z$	mean	1.0095	1.0030	1.0032	1.0149
	std dev	0.0387	0.0522	0.0741	0.1156
	CV	0.0383	0.0520	0.0738	0.1139
$\hat{\gamma}_Z$	mean	0.5058	0.4858	0.5029	0.4916
	std dev	0.0615	0.0806	0.1065	0.1727
	CV	0.1215	0.1659	0.2118	0.3512
$\hat{\omega}$	mean	1.0094	1.0029	1.0059	1.0340
	std dev	0.0395	0.0516	0.0748	0.1173
	CV	0.0392	0.0515	0.0744	0.1134
$\hat{a}$	mean	NA	2.0063	2.9974	6.2019
	std dev	NA	0.0710	0.2810	0.6037
	CV	NA	0.0354	0.0937	0.0973

Simulated Images (Scenario III, $\mu = 2$, $\gamma = 1$, size of simulation 80×80).



Examples of simulated gamma fields