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WATER RESOURCES AND THE REGIME OF WATER BODIES

Estimating Distribution Parameters of Extreme Hydrometeorological Characteristics by *L*-Moment Method

T. S. Gubareva and B. I. Gartsman

Pacific Institute of Geography, Far East Division, Russian Academy of Sciences, ul. Radio 7, Vladivostok, 690041 Russia

Abstract—*L*-moment method is briefly described and compared with classical methods of moments and maximal likelihood. Algorithms are given for calculating parameters of some distributions widely used in engineering hydrology, meteorology, etc. The advantages of *L*-moment method relative to alternative methods are analyzed for several examples.

Keywords: *L*-moments, coefficients of *L*-variation, *L*-asymmetry, *L*-kurtosis, distribution parameters, *GEV, GPD*, lognormal distribution, III-type Pearson distribution.

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INTRODUCTION

Parameter estimation for analytical probability distributions based on empirical data is an important problem in engineering hydrometeorology. For a long time, it has been actively discussed and innovated, at least, in what regards the distributions of extreme values. The main issue is due to the fact that estimating extreme characteristics involves the extrapolation of distribution laws into the domain of events with small probabilities, in many cases, far beyond the range of available observational data. Theoretically, such extrapolation, based on purely statistical approach, cannot be reliable; however, such estimates are of critical importance for economic practice.

A number of statistical methods are available for evaluating distribution parameters by sample data. Their development is in progress in probability theory and mathematical statistics, and the results are more or less actively assimilated in hydrometeorology. An important point is the identification of the perspectives of a method for the practical evaluation of parameters of various characteristics. Hereafter, the method is presented as applied to the issue currently most complicated and urgent—evaluating maximal discharges with low occurrence.

The methods of moments and maximal likelihood are in wide use in Russia. Their description can be found in textbooks on statistics, while some aspects of their application to hdyrometeorological variables are described in special publications [7, 8]. A considerable achievement of the late XX century in statistical estimation was the method of *L*-moments proposed by J.R.M. Hosking [13]. The basic methodological aspects of the method's application were published by him in co-authorship with J.R. Wallis in 1997. Nowadays, this method is in wide use for estimating various hydrometeorological variables.

Russian publications give few references to this method [2, 5, 6], and its descriptions are incomplete and insufficient for practical use. It has not been used in engineering calculation practice in Russia. The objective of this paper is to give a systematic description of the theoretical basis of *L*-moments method and to compare it with conventional methods, as well as to describe the algorithms of parameter estimation for the most frequently used distribution laws. The demonstration of the advantages of *L*-moments method is aimed primarily to extending its use by Russian researchers and engineers.

THE ADVANTAGES OF *L*-MOMENTS METHOD

The advantages of *L*-moments as compared with ordinary distribution moments are shown in several studies [13, 14, 18]. First, *L*-moments always exist where the mean value exists for the probability distribution. This extends to the cases for which ordinary higher order moments may not exist as, for example, the third and fourth moments of *GEV* distribution with a heavy tail and the distribution shape parameter $k \le -1/3$ and $k \le -1/4$ do not exist. The second central moment for *GEV* distribution does not exist when the shape parameter $k \le -0.5$. However, *L*-moments and their ratios exist in all ranges mentioned above.

Second, unlike ordinary moments, only linear functions of sample values of the variable are used to estimate *L*-moments based on an observational series. The result is that the sample estimates of *L*-moments are unbiased and more effective; they are also less sensitive to random outliers and gross observational errors than the sample estimates based on ordinary moments.

L-moments, as well as ordinary moments, are used as the first step of distribution parameter estimation procedures based on available measurement data. The second step of such procedure is to obtain the parameters based on L-moment estimates. This procedure is much more convenient than the maximum likelihood method, since it allows one to draw data on the shape of the distribution (based on L-moment estimates) and, as a rule, to obtain the parameters by relatively simple direct calculations.

The application of the maximum likelihood method is based on the assumed knowledge of the true distribution law, which significantly reduces the potentialities of its practical application. The solutions by the maximum likelihood method for most distributions used in hydrology, even in the approximate form, are systems of transcendent equations, whose solutions are sought for by numerical optimization procedures. The major difficulties here are due to the likelihood function having many local maximums and to the computational problems in the search for maximum. In some situations, the computation process is unpredictable, i.e., it can oscillate, diverge, interrupted, etc. The existence of numerous local maximums for the III-type Pearson distribution was shown in studies [12, 17, 20].

A significant fact is that in Russian regulatory documents, the maximum likelihood method is implemented in an approximate form and recommended only for Kritskii–Menkel distribution. The experience in the application of maximum likelihood method in the form recommended in SNIP 2.01.14-83 shows that it is often impossible to obtain estimates for maximal-discharge series because of computational problems [3], i.e., the only universal method for engineering–hydrological practice available in Russia is the method of moments.

L-MOMENTS OF ANALYTICAL DISTRIBUTION FUNCTIONS AND THEIR SAMPLE ESTIMATES

The brief description of the theoretical principles and algorithms of L-moments method follows the work [14]. Its authors regard the method they propose as an alternative for representing the shape of probability distribution curves and emphasize that, historically, L-moments appeared as a modification of "probability weighted moments," proposed by J. Greenwood et al. [9]. Two special cases of probability weighted moments of the rth order, α_r and β_r, exist for a distribution function represented in an integral form:

$$\alpha_r = \int_0^1 x(F)(1-F(x))^r dF, \quad \beta_r = \int_0^1 x(F)F(x)^r dF, \quad r = 0, 1, 2, \ldots, \tag{1}$$

by contrast to ordinary moments defined as

$$E(X^r) = \int_0^1 x(F)^r dF, \quad r = 0, 1, 2, \ldots, \tag{2}$$

where $x(F)$ is the quantile function inverse to the distribution function $F(x)$.

It has been shown that L-moments λ_r of a random variable X can be defined in terms of the probability weighted moments, in particular, as

$$\lambda_1 = \beta_0, \tag{3}$$

$$\lambda_2 = 2\beta_1 - \beta_0, \tag{4}$$

$$\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0, \tag{5}$$

$$\lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0, \tag{6}$$

or in the general form

$$\lambda_{r+1} = (-1)^r \sum_{k=0}^{r} p^*_{r,k}\beta_k, \tag{7}$$

where $p^*_{r,k}$ are coefficients defined as

$$p^*_{r,k} = (-1)^{r-k}\binom{r}{k}\binom{r+k}{k} = \frac{(-1)^{r-k}(r+k)!}{(k!)^2(r-k)!}. \tag{8}$$

Clearly, the first L-moment is equivalent to the expectation of the distribution. The second L-moment serves as a scale of the distribution. The L-moments of higher order, converted to this scale, were called L-moment ratios

$$\tau_r = \lambda_r/\lambda_2, \quad r = 3, 4, \ldots. \tag{9}$$

The L-moment ratios determine the shape of the distribution irrespective of the scale of the variables being measured. J.R.M. Hosking proposed the following terms: λ_1 is L-location, corresponding to the ordinary first moment or the norm; λ_2 is L-scale, analogous to the ordinary standard deviation; $\tau = \lambda_2/\lambda_1$ is L-variation coefficient ($L-Cv$), an analogue of the ordinary coefficient of variation Cv; τ_3 is L-skewness ($L-Cs$), and analogue of the ordinary coefficient of skewness Cs; τ_4 is L-kurtosis coefficient, an analogue of the ordinary coefficient of kurtosis.

In practice, we commonly deal with a sample including a finite number of observations n. First, unbiased sample estimates of probability weighted moments of distribution b_r [15] for a sample arranged in ascending order $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$ with a size of n, which can be represented as

$$b_0 = n^{-1}\sum_{j=1}^{n} x_{j:n}, \tag{10}$$

$$b_1 = n^{-1}\sum_{j=2}^{n}\frac{(j-1)}{(n-1)}x_{j:n}, \quad (11)$$

$$b_2 = n^{-1}\sum_{j=3}^{n}\frac{(j-1)(j-2)}{(n-1)(n-2)}x_{j:n}. \quad (12)$$

The general expression for b_r has the form

$$b_r = n^{-1}\sum_{j=r+1}^{n}\frac{(j-1)(j-2)\ldots(j-r)}{(n-1)(n-2)\ldots(n-r)}x_{j:n}. \quad (13)$$

By analogy with expressions (3)–(7), sample L-moments are defined as

$$l_1 = b_0, \quad (14)$$

$$l_2 = 2b_1 - b_0, \quad (15)$$

$$l_3 = 6b_2 - 6b_1 + b_0, \quad (16)$$

$$l_4 = 20b_3 - 30b_2 + 12b_1 - b_0. \quad (17)$$

In the general form

$$l_{r+1} = \sum_{k=0}^{r}p^*_{r,k}b_k; \quad r = 0, 1, \ldots, n-1, \quad (18)$$

where coefficients $p^*_{r,k}$ are defined by (8). As can be seen from the expressions given here, the sample L-moments are linear functions of unbiased estimates of probability weighted moments; therefore, they are also unbiased estimates of λ_r.

the sample L-moment ratios of the rth order are defined by the expressions

$$t = l_2/l_1, \quad (19)$$

$$t_r = l_r/l_2, \quad r = 3, 4, \ldots, \quad (20)$$

whence t is the sample coefficient of L-variation, t_3 is the sample coefficient of L-asymmetry, t_4 is the sample L-curtosis. The sample L-moment ratios are biased estimates.

PARAMETER ESTIMATION AND SHAPE ANALYSIS OF DISTRIBUTIONS BASED ON L-MOMENTS

For most distributions, formulas for parameter estimation based on$L$-moments can be found in the appropriate literature in the explicit form. Table 1 gives algorithms for mutual calculations of L-moments and parameters for several three-parameter distribution laws, characterized by shift, shape, and scale parameters. Parameter estimation based on samples can be made by substitution into these algorithms of appropriate sample estimates of L-moments and L-moment ratios.

The types of distribution laws are chosen from those used in hydrometeorology from the viewpoint of their perspective, by which we mean the possible adequacy of the description of extreme hydrological values. The mathematical form of expressions and parameter denotations are taken from [14]. Table 1 gives the distributions in generalized form, which, with certain values of parameters, pass into other known distributions.

Thus, the generalized extreme distribution *GEV* with the shape parameter $k = 0$ coincides with Gumbel distribution; with $k < 0$, it yields an extreme II-type distribution, known as Freshet distribution; while with $k > 0$, we obtain an extreme distribution of the III type (Weibull distribution). Similarly, the generalized Pareto *GPD* with $k = 0$ passes into exponential distribution; with $k = 1$, into uniform distribution, bounded by the shift value ξ from below and by $\xi + \alpha$ from above.

The lognormal distribution, parameterized by parameters of shift, shape, and scale (Table 1), in J.R.M. Hoskings' opinion, has some advantages over the accepted parameterization with the use of ordinary moments μ and σ and the shift parameter ξ. The major advantage is the more stable estimate of the parameters used here at the near-zero skewness of distributions. The generalized lognormal distribution can have either positive skewness (lower bound, the shape parameter $k < 0$) or negative skewness (upper bound, $k > 0$). With $k = 0$, the distribution coincides with normal law.

Table 3 also gives calculation formulas for Pearson III distribution of *P*III type, which is widely used both in Russia and abroad. Kritskii–Menkel distribution (three-parameter gamma distribution) has no analytical expression; the authors of [1] recommend that L-moments based on a procedure of numerical optimization of solution and the construction of special diagrams be used for this distribution.

The procedures for parameter estimation can be implemented in any computation environment. A convenient instrument is "*R*-Project for Statistical Calculations" [16], which contains practically all major components required for the construction of various procedures for statistical estimation. Of particular use is lmomco package, including parameter estimation algorithms by L-moment method for almost all distributions in use. However, the widely used Microsoft Excel environment enables practically any variant of calculations with the use of L-moments.

By now, the authors have implemented algorithms for the construction of 13 analytical curves with the use of Excel built-in mathematical functions and macrocommands, including algorithms for normal, exponential, *GEV*, *GPD*, Gumbel, two- and three-parameter lognormal, generalized logistic, *P*III, Pareto, Weibull, Frechet, and two-parameter gamma distribution. Most these distributions are in wide use in hydrometeorology. For most these distribution laws,

Table 1. Parameter calculation for distribution laws by L-moments [14] (ξ, α, k are shift, scale, and shape parameters of distributions, except for Pearson III distribution, where ξ is shift parameter, β is scale parameter, and α is shape parameter; Γ is the symbol of gamma function, Φ is integral function of standard normal distribution, G is incomplete gamma function, I is incomplete beta function; Z is a random variable with standard normal distribution; the values of coefficients $E_0 = 2.0466534$, $E_1 = -3.654437$, $E_2 = 1.8396733$, $E_3 = -0.20360244$, $F_1 = -0.20182173$, $F_2 = 1.2420401$, $F_3 = -0.21741801$)

Distribution type	Moments	Parameter
Generalized extreme *GEV* $f(x) = \alpha^{-1}e^{-(1-k)y-e^{-y}}$ $y = \begin{cases}-k^{-1}\log\{1-k(x-\xi)/\alpha\}, & k\neq 0\\ (x-\xi)/\alpha, & k = 0\end{cases}$ $x(F) = \begin{cases}\xi+\alpha\{1-(1-F)^{k}\}/k, & k\neq 0\\ \xi-\alpha\log(1-F), & k = 0\end{cases}$	For $k > 1$ $\lambda_1 = \xi+\alpha\{1-\Gamma(1+k)\}/k$ $\lambda_2 = a(1-2^{-k})\,\Gamma(1+k)/k$ $\tau_3 = 2(1-3^{-k})/(1-2^{-k})-3$ $\tau_4 = \dfrac{5(1-4^{-k})-10(1-3^{-k})+6(1-2^{-k})}{(1-2^{-k})}$	$k \approx 7.8590c + 2.9554c^2$ $c = \dfrac{2}{3+\tau_3}-\dfrac{\log 2}{\log 3}$ $\alpha = \dfrac{\lambda_2 k}{(1-2^{-k})\Gamma(1+k)}$ $\xi = \lambda_1-\alpha\{1-\Gamma(1+k)\}/k$
Generalized Pareto*GPD* $f(x) = \alpha^{-1}e^{-(1-k)y}$ $y = \begin{cases}-k^{-1}\log\{1-k(x-\xi)/\alpha\}, & k\neq 0\\ (x-\xi)/\alpha, & k = 0\end{cases}$ $x(F) = \begin{cases}\xi+\alpha\{1-(1-F)^{k}\}/k, & k\neq 0\\ \xi-\alpha\log(1-F), & k = 0\end{cases}$	$\lambda_1 = \varepsilon+\alpha/(1+k)$ $\lambda_2 = \alpha/\{(1+k)(2+k)\}$ $\tau_3 = (1-k)/(3+k)$ $\tau_4 = \dfrac{(1-k)(2-k)}{(3+k)(4+k)}$	$k = (1-3\tau_3)/(1+\tau_3)$ $\alpha = (1+k)(2+k)\lambda_2$ $\xi = \lambda_1-(2+k)\lambda_2$
Generalized lognormal $f(x) = \dfrac{e^{ky-y^2/2}}{\alpha\sqrt{2\pi}}$, $y = \begin{cases}\dfrac{\log\{1-k(x-\xi)/\alpha\}}{-k}, & k\neq 0\\ (x-\xi)/\alpha, & k = 0\end{cases}$ $X = \begin{cases}\xi+\alpha(1-e^{-kZ})/k, & k\neq 0\\ \xi+\alpha Z, & k = 0\end{cases}$	$\lambda_1 = \xi+\alpha(1-e^{k^2/2})/k$ $\lambda_2 = \dfrac{\alpha}{k}e^{k^2/2}\{1-2\Phi(-k/\sqrt{2})\}$ $\tau_3 \approx -k\dfrac{A_0+A_1k^2+A_2k^4+A_3k^6}{1+B_1k^2+B_2k^4+B_3k^6}$ $\tau_4 \approx \tau_4^0+k^2\dfrac{C_0+C_1k^2+C_2k^4+C_3k^6}{1+D_1k^2+D_2k^4+D_3k^6}$	$k = -\tau_3\dfrac{E_0+E_1\tau_3^2+E_2\tau_3^4+E_3\tau_3^6}{1+F_1\tau_3^2+F_2\tau_3^4+F_3\tau_3^6}$ $\alpha = \dfrac{\lambda_2 k e^{-k/2}}{1-2\Phi(-k/\sqrt{2})}$ $\Phi(x) = \int_{-\infty}^{x}\left[(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}x^2\right)\right]d(x)$ $\xi = \lambda_1-\dfrac{\alpha}{k}(1-e^{k^2/2})$
Pearson III $f(x) = \dfrac{(x-\xi)^{\alpha-1}e^{-(x-\xi)/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$ $F(x) = G\left(\alpha, \dfrac{x-\xi}{\beta}\right)/\Gamma(\alpha)$ $G(\alpha, x) = \int_0^x t^{\alpha-1}e^{-t}dt$	$\lambda_1 = \xi+\alpha\beta$ $\lambda_2 = \pi^{-1/2}\beta\Gamma(\alpha+1/2)/\Gamma(\alpha)$ $\tau_3 = 6I_{1/3}(\alpha, 2\alpha)-3$ $I_x(p, q)$ $= \dfrac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\int_0^x t^{p-1}(1-t)^{q-1}dt$ if $a \geq 1$, $\tau_4 \approx \dfrac{C_0+C_1\alpha^{-1}+C_2\alpha^{-2}+C_3\alpha^{-3}}{1+D_1\alpha^{-1}+D_2\alpha^{-2}}$ if $a < 1$, $\tau_4 \approx \dfrac{1+G_1\alpha+G_2\alpha^2+C_3\alpha^3}{1+H_1\alpha+H_2\alpha^2+H_3\alpha^3}$	if $0 < \lvert\tau_3\rvert < 1/3$, $\alpha \approx \dfrac{1+0.2906z}{z+0.1882z^2+0.0442z^3}$ $z = 3\pi\tau_3^2$ if $1/3 \leq \lvert\tau_3\rvert < 1$, $\alpha = \dfrac{0.36067z-0.59567z^2+0.25361z^3}{1-2.78861z+2.56096z^2-0.77045z^3}$ $z = 1-\lvert\tau_3\rvert$

three methods of parameter estimation by sample were implemented: the methods of moments, L-moments, and maximum likelihood. The example in Fig. 1 gives different variants of empirical and analytical distribution curves.

Examples of calculated L-moments and their ratios— L-variation, L-skewness, L-curtosis—for several series of maximal discharges are given in Table 2. Estimates of L-moments can be used, in particular, in the substantiation some distribution law as an approximation [21]. For this purpose, sample estimates of L-skewness and L-curtosis for an observational series are used as coordinates of point (t_3, t_4) in the field of L-moment diagram, which represent a series of dependencies $\tau_4 = f(\tau_3)$ for different theoretical distributions. L-moment diagram, constructed in the variation ranges $0 < \tau_3 < 0.7$ and $0 < \tau_4 < 0.6$, which are of greatest practical interest, is given in Fig. 2.

Two-parametric distributions are represented in the diagram by a point. Three-parametric distributions, characterized by the parameters of shape, shift, and scale, are represented in the diagram by curves, which can be approximated by the polynomial function [14]

$$\tau_4 = \sum_{k=0}^{8} A_k \tau_3^k, \quad (21)$$

where A_k are coefficients, which are different for different distribution laws (Table 3).

The position of the point with coordinates (t_3, t_4) for the observational series under consideration in the diagram field in Fig. 2 may suggest which distribution law will be appropriate as an approximation of the empirical distribution. In this case, the diagram presents calculated values for maximal discharge series given in Table 2.

Table 2. L-moments of maximal discharge series

River–point	Length, years	l_1	l_2	t	t_3	t_4
Ussuri R., Koksharovka Settl.	54	785	384	0.49	0.51	0.34
Arsen'evka R., Yakovlevka V.	62	599	348	0.58	0.53	0.29
Malinovka R., Rakitnoe V.	69	504	259	0.51	0.47	0.33
Bira R., Lermontovka V.	52	300	203	0.67	0.65	0.43
Avvakumovka R., Vetka Settl.	75	386	232	0.60	0.52	0.32
Margaritovka R., Margaritovo V.	58	357	210	0.59	0.55	0.38

Table 3. Coefficients of polynomial approximation of functions $\tau_4 = f(\tau_3)$ (dash means the absence of coefficients)

Coefficient	Distribution			
	GEV	*GPD*	*LN*3	*P*III
A_0	0.10701	0	0.12282	0.12240
A_1	0.11090	0.20196	–	–
A_2	0.84838	0.95924	0.77518	0.30115
A_3	–0.06669	–0.20096	–	–
A_4	0.00567	0.04061	0.12279	0.95812
A_5	–0.04208	–	–	–
A_6	0.03763	–	–0.13638	–0.57488
A_7	–	–	–	–
A_8	–	–	0.11368	0.19383

EXAMPLES OF USE OF L-MOMENTS IN CALCULATIONS

As mentioned above, an important advantage of the method is, first, the unbiasedness of the L-moments calculated from the sample and the weak biasedness of L-moment ratios for different sample series length. Estimates of t_r and t for moderate-length and long samples have very small bias and variations; for example, according to data of [13, 14], the asymptotic bias t_3 for Gumbel distribution is $0.19n^{-1}$, and that for the normal distribution t_4 is $0.03n^{-1}$. The bias in small samples can be estimated by modeling; however, in the general case for the sample size of 20 and more, the bias of the coefficient of L-variation is small.

It is also important that the extreme (large or small) terms of sample, containing important information about the tails of theoretical distribution, have small weight in L-moment parameter estimates. Such estimates are stable, which is in principle characteristic of the maximum likelihood method. The property of stability can be seen even from the comparison of formulas for evaluating simple moments and L-moments.

For sample estimates, this property manifests itself in the degree of change in their values caused by elimination of extreme values from the series.

Let us consider an example of a series of maximal annual discharges in the Zima River at Zulumai Settlement (57 years) with characteristics of $Cv = 0.77$, $Cs = 2{,}89$, $L - Cv = 0.35$, and $L - Cs = 0.37$. Once the extreme (maximal) value is excluded, Cv and Cs will decrease to 0.62 and 1.66, respectively; and $L - Cv$ and $L - Cs$, to 0.31 and 0.28, respectively; expressed in percent, the decrease will be 20% in Cv, 42% in Cs, 10% in $L - Cv$, and 25% in $L - Cs$. As shown in [13], the ordinary coefficient of skewness in samples with size of 5000 and more can be largely determined by its extreme terms, while the coefficient $L - Cs$ will not.

One more example is the series of maximal discharges of rain freshets in the Avvakumovka River at

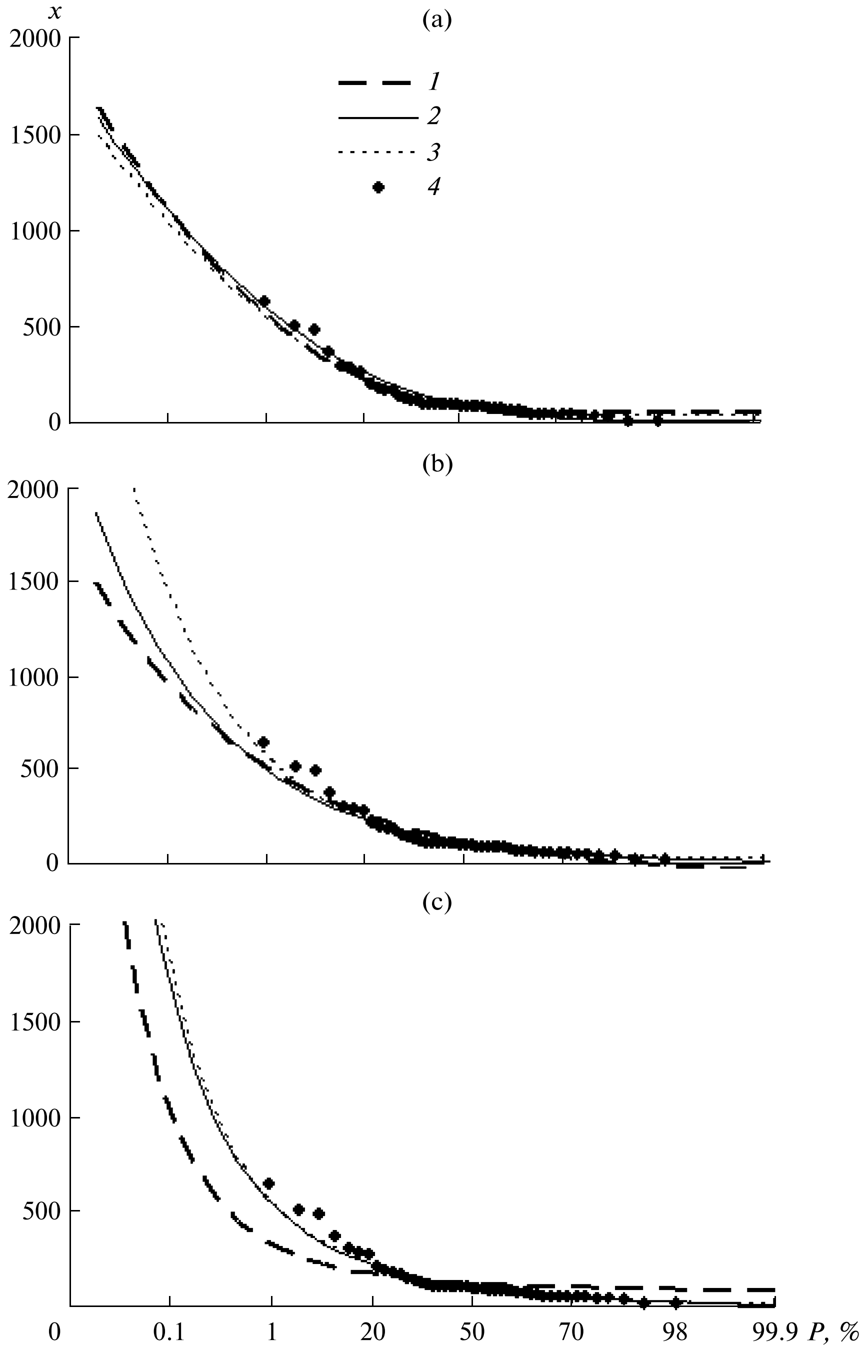

Fig. 1. Examples of analytical distributions of maximal annual discharges in the Izvilinka R. at Izvilinka V. (a) *P*III, (b) *LN3*, (c) *GEV*, whose parameters were calculated by (*1*) moments method, (*2*) maximum likelihood, (*3*) *L*-moments; (*4*) empirical curve.

Vetka Settlement (75 years in length). The largest value in the series exceeds the second largest by 1.6 times. We for subsamples with a length of 20 to 74 by random elimination of terms from the initial series and calculate the skewness and *L*-skewness coefficients for each subsample. Standardized results of calculations are given in Fig. 3, where sample estimates of parameters show changes of different character with increasing number of terms in the sample. Overall, *Cs* varies from 0.5 to 1.5; whereas t_3, from 0.8 to 1.4; the variance of the series of calculated *Cs* and t_3 (the total number of subsamples is 55) is 0.046 and 0.009, respectively. The variance of *Cs* estimates for subsamples from 20 to 46 years is 0.078, and that for subsamples from 47 to 74 years is 0.016; while the variance of t_3 estimates is 0.017 and 0.003, respectively.

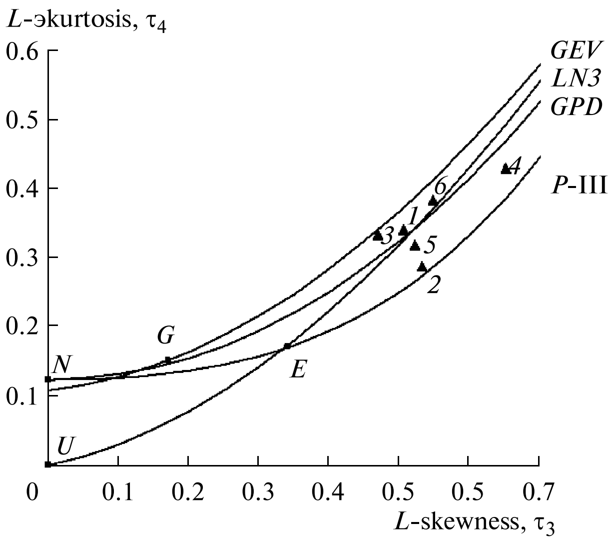

Fig. 2. Diagram of L-moments ratios. Two-parameter distributions: U is uniform, N is normal, G is Gumbel, E is exponential; three-parameter distributions: *GEV* is generalized extreme, *LN3* is lognormal, *GPD* is generalized Pareto, *P*III is Pearson III type. (*1–6*) Points, corresponding to river numbers in Table 2.

The visual analysis of plots in Fig. 3 shows insignificant bias and relatively rapid decrease in the variance of skewness estimate by L-moments method with increasing sample length n relative to the ordinary method of moments. For larger subsamples, the value of L-skewness coefficient remains almost unchanged, unlike the ordinary *Cs*, whose value depends on whether an outlier enters the subsample. According to data [14], when the deviation of the outlier is very large, the diagram of estimates by the method of moments distinctly separates into two fields of points, while this is not the case for L-moments estimates. This suggests the relative stability of L-moments estimates for small samples.

Let us analyze an example of correlating the main method for parameters' assessment on mass, relatively homogeneous material. The analysis involves 36 series of maximal annual river discharges in Primorskii Krai with a length of 40 to 76 years, prechecked for the absence of statistically significant linear trends. The rivers of this region feature the predominance of rain summer freshets in their regime and high specific values of maximal freshet discharges, large variance and skewness values for maximal runoff series. The parameter estimation methods are compared on *P*III, *GEV*, and *LN*3 distributions. The two latter distributions are most adequate for the conditions under consideration (among distributions that are in wide use in hydrology), as can be seen from both the analysis of vast literature in this field and the results of the authors' own studies [10, 11].

Three different criteria are used to assess the approximation quality. The first is the χ^2 criterion widely used for the analysis of the agreement between sample data and the given distribution law. The other two criterions, denoted by ω and S, were used by the authors before to compare calculation schemes against mass data on maximal discharges of Northern Eurasia rivers.

The criterion omega is used in accordance with recommendations of Yu.B. Vinogradov and represent the total relative deviation between the abscissas of the empirical and analytical occurrence curves. It is calculated as

$$\omega = \sum_{i=1}^{n} \left(\frac{|p^* - p^{**}|}{p_b^* - p_a^*} \right)_i, \tag{22}$$

where p^* is the empirical occurrence, p^{**} is analytical occurrence, p_a^*, p_b^* are the boundaries of the confidence interval of empirical occurrence; all these values refer to each ith value from the observation sample with the total volume of n. Individual formulas [2, p. 241] are recommended for evaluating p_a^*, p_b^*. The minimal value of criterion ω corresponds to the best approximation quality by probability.

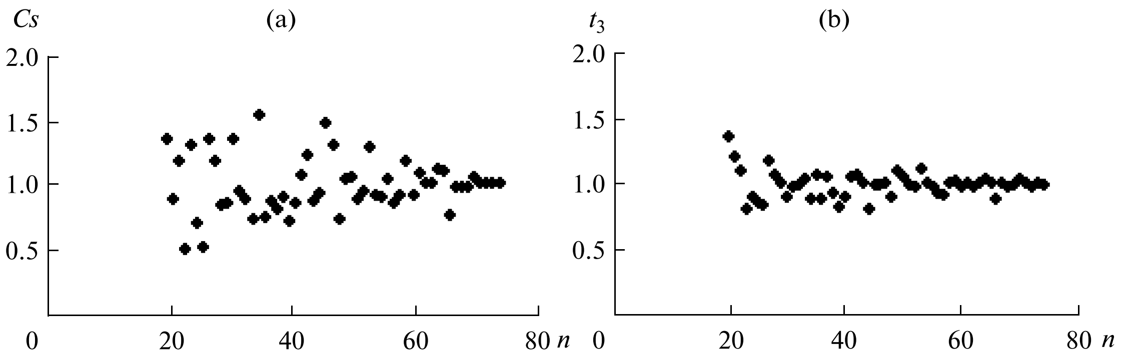

Fig. 3. Variations in (a) skewness and (b) L-skewness coefficients vs. the number of terms in the subsample. An example of the Avvakumovka R., Vetka Settl.

Table 4. Comparison of approximation quality of sample data in assessing distribution law parameters by different methods (MM is method of moments, *L*-M is *L*-moments method, ML is maximum likelihood method)

Estimation characteristic	Distribution								
	P III			*LN*3			*GEV*		
	MM	*L*-M	ML	MM	*L*-M	ML	MM	*L*-M	ML
The number of cases where null hypothesis was accepted by χ^2 criterion (5% confidence level)	23	33	34	18	35	35	8	35	35
The number of cases with the best value of criterion ω	6	24	6	2	13	21	4	10	22
Mean value of criterion ω	20.4	11.4	17.0	11.8	6.06	6.01	14.1	6.36	7.60
The number of cases with the best value of criterionў *S*	6	22	8	3	24	9	–	20	16
Mean value of criterion *S*	3.27	3.28	4.63	3.98	2.73	13.2	13.8	3.53	72.8

Criterion *S* is an estimate by the least square method, and its minimal value corresponds to the best approximation quality by the values of the variable. It is calculated by the well-known formula

$$S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(k_p^* - k_p^{**})_i^2}, \quad (23)$$

where k_p^* and k_p^{**} are quantiles, with the same occurrence, of the empirical and analytical distribution curves, respectively.

The characteristics used in the case of comparison by 36 samples for each of the three methods included the number of cases when the zero hypothesis was accepted by χ^2 criterion with 5% significance level; the number of cases with the best values of ω and *S* criteria; the mean weighted values of ω and *S* criteria. The results of comparison are given in Table 4. They show the generally comparable quality of the results of applying *L*-moments and maximum likelihood method and the much worse results of the ordinary moment method.

CONCLUSIONS

This paper briefly describes the up-to-date method of statistical parameter estimation of probability distributions—the *L*-moments method. Algorithms for the calculation of *L*-moments estimates by observational series are given. Relationships between alternative characteristics of distribution shape ($L - Cv$, $L - Cs$, and *L*-kurtosis) with the parameters of distribution functions, and the algorithms for determining parameters by *L*-moments method for a number of distribution functions used in the practice of engineering–hydrological calculations are described. The advantages of the method are shown: the existence of *L*-moments for some variants of distributions with heavy tails, for which the ordinary moments do not exist; the low bias and efficiency of *L*-moment estimates; the possibility to substantiate the type of the distribution law by *L*-asymmetry and *L*-kurtosis.

The comparison of the *L*-moments method with the maximum likelihood method shows that the small bias, the consistency and efficiency of parameter estimates are the common advantages of both methods. However, *L*-moment method can be implemented by relatively simple and more stable computation procedures and a priori does not require one to know the actual distribution law. As compared with the ordinary method of moments, the *L*-moments method has considerable advantages regarding all major requirements imposed to distribution parameter estimates. A long-run study perspective is the analysis of coordinate estimates for distribution curves of hydrometeorological variables obtained with the help of this method.

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