Statistics - Homework 6 (Due Dec. 20, 2017)

1. Let X be a random variable with the following density:

$$f_X(x) = \tfrac{b}{a}\left(\frac{x-m}{a}\right)^{b-1} e^{-\left(\frac{x-m}{a}\right)^b}, \ (a > 0,\ b > 0,\ x \ge m).$$

[Weibull distribution or the Extreme Value type III (EV3) distribution.]

(1) Let $Y = \left(\dfrac{X-m}{a}\right)^b$. Find the density function of Y.

(2) Show that the cumulative distribution function of X can be expressed as

$$F_X(x) = 1 - e^{-\left(\frac{x-m}{a}\right)^b}.$$

(3) Generate a random sample of size 1,000 from a Weibull distribution with parameters $m = 275.31,\ a = 13900,\ b = 2.9114$ and plot its empirical CDF.

(4) Calculate the sample coefficient of skewness of the random sample in (3).

2. Let $(x_1,\ x_2,\cdots,x_n)$ be a random sample of size n from a distribution with a cumulative distribution function $F_X(\cdot)$ and $x_1 < x_2 < \cdots < x_n$. The empirical cumulative distribution function (ECDF) based on the random sample is defined as

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x]}(x_i);\ -\infty < x < \infty .$$

Prove the following results:

(1) $F_n(X)$ is asymptotically normal, i.e., $F_n(X)$ approaches a normal distribution as n approaches infinity.

(2) $E[F_n(X)] = F_X(X)$

(3) $Var[F_n(X)] = \dfrac{1}{n} F_X(X)[1 - F_X(X)]$

(4) $\sqrt{n}[F_n(X) - F_X(X)]$ approaches a normal distribution with mean 0 and variance $F_X(X)[1 - F_X(X)]$ as n approaches infinity.

3. A random sample of size n (i.e., $x_1,\ x_2,\cdots,x_n$) from an exponential density $f_X(x;\ \lambda) = \lambda e^{-\lambda x}$ is available.

(1) Prove that the maximum likelihood estimate of λ is $\hat{\lambda}_{MLE} = \dfrac{n}{\sum_{i=1}^{n} x_i} = \dfrac{1}{\bar{x}_n}$.

(2) Prove that the above maximum likelihood estimator is (or is not) an unbiased estimator. *Hint*: The sample mean of a random sample from an exponential density has a gamma distribution. [Note: $E\left[\dfrac{1}{X}\right] = \dfrac{1}{E[X]}$ is not generally true.]

4. A random sample of size n_1 from a normal distribution with mean μ_1 and standard deviation σ_1 is available. Another random sample of size n_2 from a normal distribution with mean μ_2 and standard deviation σ_2 is also available. The two random samples are independent.

(1) Find a maximum likelihood estimator of $\theta = \mu_1 - \mu_2$.

(2) If the sum of n_1 and n_1 $(n = n_1 + n_2)$ is fixed, how should the sizes of the two random samples be chosen in order to minimize the variance of the maximum likelihood estimator in (1).

5. Let $(x_1,\ x_2,\cdots,x_n)$ be a random sample from an exponential density $f_X(x;\ \lambda)=\lambda e^{-\lambda x}$ and $(y_1,\ y_2,\cdots,y_m)$ be a random sample from another exponential density $f_Y(y;\ \theta)=\theta e^{-\theta y}$ with $\theta=2\lambda$. The two random samples are independently observed. Find the maximum likelihood estimate of λ.
6. Let $X_1,\ X_2,\cdots,X_n$ be independently and identically distributed (IID) from a continuous uniform distribution $U[0,\ \theta]$.
   (1) Find a maximum likelihood estimator of θ, and then find the mean and mean-squared error of your estimator.
   (2) Find a method of moments estimator of θ, and then find the mean and mean-squared error of your estimator.
7. Let $(x_1,\ x_2,\cdots,x_n)$ be a random sample from a distribution with probability density $f_X(x;\ \theta)=\dfrac{\theta 2^{\theta}}{x^{\theta+1}},\ x\ge 2,\ \theta>1$.
   (1) Find the method of moments estimate of θ.
   (2) Find the maximum likelihood estimate of θ.
8. The radius (r) of a circle is measured with an error of measurement which is normally distributed with zero mean and unknown variance σ^2. Given the following independent measurements of the radius:

51.07	52.49	48.07	48.94	49.33
50.34	51.30	40.33	41.38	44.05
51.45	48.89	56.42	49.21	49.41
46.03	53.64	41.13	46.17	57.07

Find an unbiased estimator of the area of the circle and use the above data to calculate an unbiased estimate of the circle area.