STATISTICS Midterm Exam (Nov. 8, 2016)

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NOTE: You need to provide detailed calculations or explanations. Do NOT just give a magic number without showing any calculations or explanations.

1. Let X and Y be two *independent* random variables and let $P(Y > X)$ represent the probability that a random number (y) from Y is larger than a random number (x) from X.

   (1) Prove that $P(Y > X) = \int_{-\infty}^{+\infty} (1 - F_Y(x)) f_X(x) dx$. **[8%]**

   $$P(Y > X) = \int_{-\infty}^{\infty} P(Y > x) f_X(x) dx = \int_{-\infty}^{\infty} \int_{x}^{+\infty} f_Y(y) dy f_X(x) dx$$
   $$= \int_{-\infty}^{+\infty} (1 - F_Y(x)) f_X(x) dx$$

   (2) Let X and Y be independent normal random variables of the same density function, i.e., $\mu_X = \mu_Y$ and $\sigma_X = \sigma_Y$. Calculate $P(Y > X)$ using the equation in (1). **[6%]**

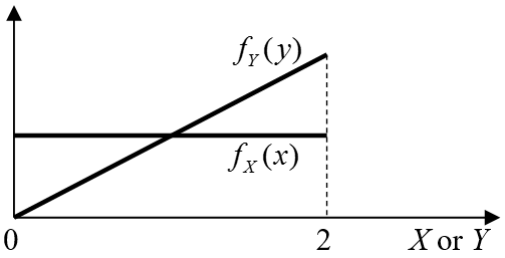
   $F_X(x) = F_Y(x)$

   $$P(Y > X) = \int_{-\infty}^{+\infty} (1 - F_Y(x)) f_X(x) dx = \int_{-\infty}^{+\infty} (1 - F_X(x)) f_X(x) dx$$
   $$= E[1 - F_X(X)] = 1 - E[F_X(X)] = 1 - E(U) \text{ (U has a uniform distribution between } [0,1]$$
   $$= \frac{1}{2}$$

   Or,

   $$P(Y > X) = \int_{-\infty}^{+\infty} (1 - F_Y(x)) f_X(x) dx = \int_{-\infty}^{+\infty} (1 - F_X(x)) f_X(x) dx$$
   $$= \int_{-\infty}^{+\infty} (1 - F_X(x)) dF_X(x) = 1 - \left[\frac{(F_X(x))^2}{2} \right]_{-\infty}^{\infty} = 1 - \frac{1}{2} = \frac{1}{2}$$

   (3) For two independent random variables X and Y of the following density functions, calculate $P(Y > X)$. **[6%]**



   $$f_X(x) = \frac{1}{2} I_{[0,2]}(x)$$

   $$f_Y(y) = \frac{y}{2} I_{[0,2]}(y), \quad F_Y(y) = \frac{y^2}{4} I_{[0,2]}(y)$$

   $$P(Y > X) = \int_{-\infty}^{+\infty} (1 - F_Y(x)) f_X(x) dx = \frac{1}{2} \int_0^2 \left(1 - \frac{x^2}{4}\right) dx = \frac{2}{3}$$

2. The probability of success for a Bernoulli trial is 0.3.

   (1) What is the probability of getting the first success at the 10th trial? **[6%]**

   Geometric distribution

   $$P(X = 10) = (1 - 0.3)^9 0.3 = 0.01210608$$

   (2) What is the probability of having the 3rd success at the 10th trial? **[6%]**

   Negative binomial distribution

   $$C(9,2)(1 - 0.3)^7 (0.3)^3 = 0.080048$$

   (3) What is the average number of trials needed to obtain the 3rd success? **[8%]**

Expected value of the negative binomial distribution

$E(X) = r / p = 3/0.3 = 10$.

3. Answer the following questions with reference to the R code and simulation.

```
> a=□
> b=□
> x1=rnorm(1,a,b)
> u1=pnorm(x1,a,b)
> c(u1,x1)
[1]  0.3136909 50.2916925
> x2=qnorm(0.8,a,b)
> x2
[1] 76.83242
```

```
> lambda=□
> u=pexp(7.12,lambda)
> punif(u,0,2)
[1] 0.3796255
```

(1) a = ? **[6%]**

(2) b = ? **[6%]**

$\Phi(z_1) = 0.3136909$, $z_1 = -0.4854$, $\dfrac{50.2916925 - a}{b} = -0.4854$.

$\Phi(z_2) = 0.8$, $z_2 = 0.8416$, $\dfrac{76.83242 - a}{b} = 0.8416$.

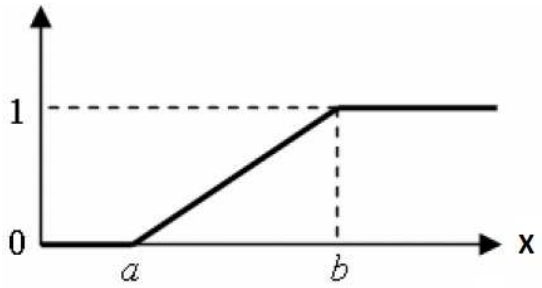
$a = 60$, $b = 20$.

(3) lambda = ? **[8%]**

$u = 2 \times 0.3796255 = 0.759251$, $1 - e^{-7.12\lambda} = 0.759251$.

$\lambda = \ln(1 - 0.759251)/(-7.12) = 0.2$.

4. The cumulative distribution function of a continuous random variable X is shown in the following figure. **[20%]**



If a+b=10 and Var[X] = 4/3, calculate E[X] and P[2 <X <5] .

$f_X(x) = \dfrac{1}{b-a} I_{[a,b]}(x)$, $Var(X) = \dfrac{(b-a)^2}{12} = \dfrac{4}{3} = \dfrac{(10-2a)^2}{12}$, $10 - 2a = \pm 4$, $a = 3$.

$E(X) = \dfrac{a+b}{2} = 5$, $P(2 < X < 5) = \dfrac{1}{2}$.

5. Taiwan is most vulnerable to natural disasters such as typhoons and earthquakes. On average, there are 3.6 typhoons passing through Taiwan annually. The average number of occurrences of earthquakes exceeding category 6 of Richter scale (芮氏地震等級) is 2.8 occurrences annually. Occurrences of typhoon and earthquakes are considered to be independent. What is the probability of having more than six natural disasters of either typhoons or earthquakes exceeding category 6 of Richter scale in one year? **[20%]**

$X_1 \sim Poisson(3.6)$, $X_2 \sim Poisson(2.8)$, $Y = X_1 + X_2 \sim Poisson(6.4)$

$1 - ppois(6, 6.4) = 0.4577$.

標準常態分布之累積機率 **(Cumulative probability of the standard normal distribution)**

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	**.5359**
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	**.5753**
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	**.6141**
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	**.6517**
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	**.6879**
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	**.7224**
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	**.7549**
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	**.7852**
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	**.8133**
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	**.8389**
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	**.8621**
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	**.8830**
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	**.9015**
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	**.9177**
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	**.9319**